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ABSTRACT

The purpose of this teaching unit is to approach elementary probability problems in three ways. First, actual trials are performed and results are recorded. Second, a simple computer simulation of the problem provided on diskette and written for Apple IIe and IIc computers, is run several times. Finally, the mathematical solution of the problem is presented. Comparison of the three sets of results--actual trials, computer simulation, and mathematical solution--provides a surprising and motivating reinforcement for students. They become involved in problem solving through hands-on trials and computer simulations. The elegance of the mathematical solutions is contrasted with the awesome number-crunching ability of the computer and the "doing is believing" experience of the actual trials. The unit can be presented during consecutive mathematics or computer science classes, or individual problems can be presented alone as a diversion in a traditional curriculum. Most of the problems are appropriate for classes from seventh grade through the college level. Appendixes include: an explanation of basic probability terms; a brief history of probability; a microcomputer unit on generating random numbers; a description of the "Monte Carlo Method"; an annotated bibliography; and a list of other references. (PK)

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PROBABILITY

Actual Trials, Computer Simulations, and Mathematical Solutions

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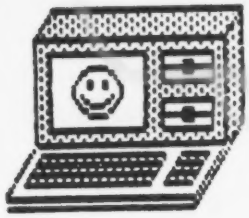
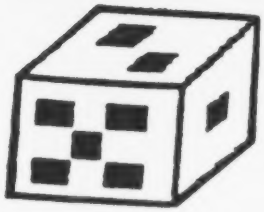
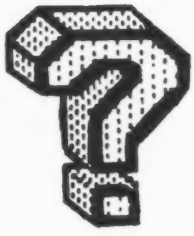




PROBABILITY: ACTUAL TRIALS, COMPUTER SIMULATIONS, AND MATHEMATICAL SOLUTIONS

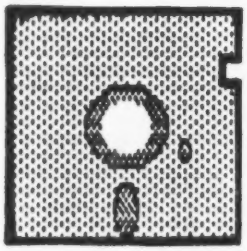


BY

KAREN DOYLE WALTON

AND

J. DOYLE WALTON



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PROBABILITY: ACTUAL TRIALS, COMPUTER SIMULATIONS,

AND MATHEMATICAL SOLUTIONS

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I. PREFACE

The material presented in this teaching unit was originally developed for a 1985 summer program entitled Computer Applications for High School Mathematics Curricula. As director of that project, sponsored by the Pennsylvania Department of Education Chapter 2 Office, I am indebted to Eugene Urbanski and John Resetar for their efficient support. I am grateful to the many students and teachers who provided feedback while performing these activities at the 1986 NCTM Annual Convention, the 1986 Pennsylvania Council of Teachers of Mathematics Annual Convention, the Association of Mathematics Teachers of New York State Annual Summer Workshop at SUNY Oswego, the in-service workshops for Abington (Pa.) Junior High School teachers (under the direction of Anne Hagan Bacon), and in probability and statistics classes at Allentown College of St. Francis de Sales.

Many of the programs in this unit were either written or adapted by Betty Collis of the University of Victoria, British Columbia, from the sources listed in the Bibliography or References. REM statements at the beginning of those programs indicate the original source of a modified program. Dr. Collis generously allowed me to copy her disk of probability problems for dissemination at various workshops. Other problems and programs are the work of the authors listed in the Bibliography, whose contributions to this project were considerable.

I am appreciative of the time and interest invested by my son (and coauthor) Doyle Walton, who performed the actual trials and prepared transparencies for many of the experiments included in this unit for his Pennsylvania Junior Academy of Science Project. But my greatest appreciation is due Daniel G. Gambet, OSFS, President, and Alexander T. Pocetto, OSFS, Vice President for Academic Affairs, Allentown College, for generously providing opportunities for me to continue my research, writing, and teaching.

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II. INTRODUCTION

Many students regard the real world, computers, and pure mathematics as disjoint. The purpose of this teaching unit is to approach elementary probability problems in three ways. First, actual trials are performed and results are recorded. Second, a simple computer simulation of the problem (provided on diskette and written for Apple IIe and IIc computers) is run several times. Finally, the mathematical solution of the problem is presented. Comparison of the three sets of results--actual trials, computer simulation, and mathematical solution--provides a surprising and motivating reinforcement for students. They become "involved" in problem solving through hands-on trials and computer simulations. The elegance of the mathematical solutions is contrasted with the awesome number-crunching ability of the computer and the "doing is believing" experience of the actual trials.

The unit can be presented during consecutive mathematics or computer science classes, or individual problems can be presented alone as a diversion in a traditional curriculum. Most of the problems are appropriate for classes from seventh grade through the beginning college level.

III. PROCEDURE

Actual Trials

The supplies needed to perform the problems are simple and inexpensive. The number of actual trials performed is determined by the problem and the classroom situation. Ideally, groups of

two or three students perform the trials, tallying results, which are then combined to produce totals for the class as a whole. If necessary, larger groups and fewer trials can suffice.

Computer Simulations

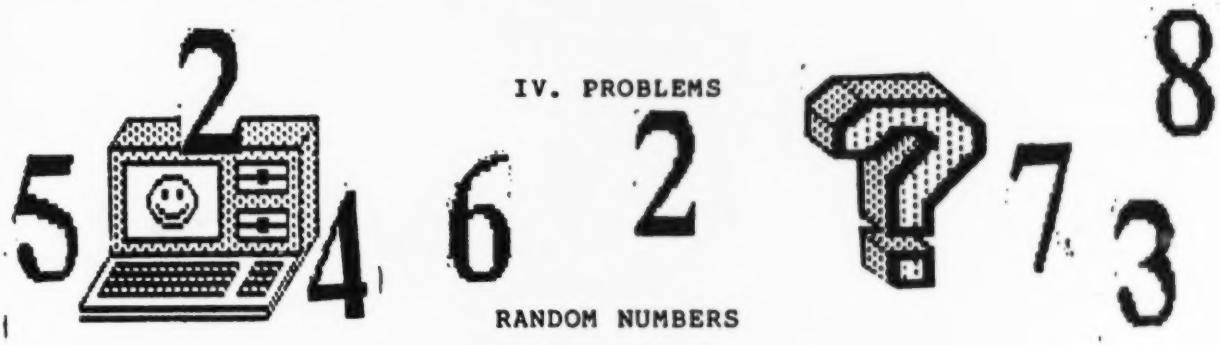
Again the ideal situation is small groups of students running the programs on Apple IIe or IIc computers. (The programs can be easily modified to run on other computers.) However, each of the problems has been presented successfully in one-computer classrooms. As with the actual trials, the number of computer runs can be varied according to time constraints, the nature of the problem, and the desired degree of accuracy.

The time spent discussing the computer program can vary depending on the backgrounds and educational objectives of the teacher and students. The unit can be used successfully with no reference to the programs. Preferably, students with some background in elementary programming can be asked to LIST the program and observe, for example, which line(s) "flips the coin" or "draws the bill." More advanced computer students can modify the given programs or write their own. The highly motivated student can be encouraged to identify a new problem and solve it in the prescribed format of actual trials, computer simulation, and mathematical solution.

Mathematical Solutions

The mathematical solutions presented assume no previous knowledge of elementary probability theory on the part of the student. The learning experience is enhanced if the teacher has

the background in probability theory presented in Appendix A, entitled "Basic Probability Terms." Use of these terms in discussing the problems facilitates communication without emphasizing formal definitions.



IV. PROBLEMS

RANDOM NUMBERS

EXPERIMENT: Generate randomly the digits 0, 1, 2, 3,..., 9 for a total of 100 digits. How many 0s, 1s, 2s,..., 9s occur?

ACTUAL TRIALS: Label ten slips of paper 0, 1, 2,..., 9 and put them into a bag. Draw one slip, record the number, and return the slip to the bag. Repeat the experiment 100 (200? 300?) times, tallying your results. Record the total number of 0s drawn, the number of 1s drawn,..., the number of 9s drawn.

COMPUTER SIMULATION: RUN RANDOM NUMBERS

Suggestions:

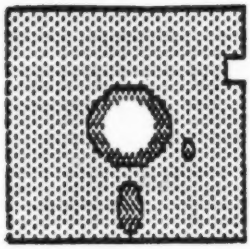
1. On the first run, generate 100 numbers (200? 300?, the same number as your actual trials).

2. Break the program by simultaneously pressing the Control and C keys and then the Return key. Type LIST 200,300. Determine which line in the program generates the random number.

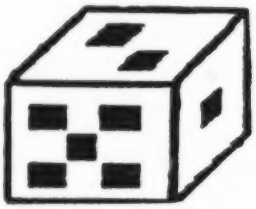
MATHEMATICAL SOLUTION: If the numbers are generated in a truly random manner, equal numbers of each digit should occur. If 10

digits (0, 1, 2,..., 9) are generated for a total of 100 digits, each number should occur 10 times (100/10 = 10).

An excellent discovery approach to studying the random generator function (RND) is "Microcomputer Unit: Generating Random Numbers" by William E. Haigh (1986), which appeared in the Mathematics Teacher. The grade level is 6-12 and the completion time is one fifty-minute period on the computer. If time permits, this worksheet (which appears in Appendix C) should be completed by the class as an introduction to the teaching unit.



THROW ONE DIE



EXPERIMENT: One die is thrown many times. What is the probability that a 1 will occur? A 2 will occur? A 3 will occur?...A 6 will occur?

ACTUAL TRIALS: Throw a die 100 (500? 1000?) times. Tally the number of 1s, 2s,..., 6s that occur. Calculate the relative frequency of 1s by dividing the number of 1s by 100 (the number of trials). Do the same for 2s, 3s,..., 6s.

COMPUTER SIMULATION: RUN THROW ONE DIE

Suggestions:

1. On the first run, let N = 1000 and M = 50. Since N represents the total number of trials in the simulation, the relative frequency of each event can be calculated easily by dividing by 1000.

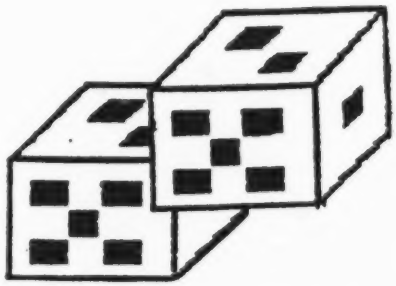
2. Break the program by pressing Control C and then the Return key. Type LIST 300,400. Determine which line in the program "tosses the die."

MATHEMATICAL SOLUTION: Sample space = S = {1, 2, 3, 4, 5, 6}

EVENT = E	N(E)	N(E)/N(S) = P(E)
E_1 = {1}	1	1/6 ≐ .166
E_2 = {2}	1	1/6 ≐ .166
E_3 = {3}	1	1/6 ≐ .166
E_4 = {4}	1	1/6 ≐ .166
E_5 = {5}	1	1/6 ≐ .166
E_6 = {6}	1	1/6 ≐ .166

Compare your results for the actual trials, computer simulation, and mathematical solution.

THROW TWO DICE



EXPERIMENT: A pair of dice is thrown and the sum of the dice is observed. If the experiment is repeated many times, what is the probability that the sum of 2 will occur? The sum of 3?...The sum of 12?

ACTUAL TRIALS: Throw a pair of dice 100 (500? 1000?) times and observe the sum. Tally the number of times the sum of 2 occurs, the sum of 3 occurs,..., the sum of 12 occurs. Calculate the relative frequency of 2s by dividing the number of 2s by 100 (the number of throws). Do the same for the sum of 3, the sum of 4,..., the sum of 12.

COMPUTER SIMULATION: RUN THROW TWO DICE

Suggestions:

1. On the first run, let N = 1000 and M = 50. Since N represents the total number of trials in the simulation, the relative frequency of each event can be easily calculated by dividing by 1000.

2. Break the program by pressing Control C and then the Return key. Type LIST 250,350. Determine which lines in the program "toss the dice."

MATHEMATICAL SOLUTION:

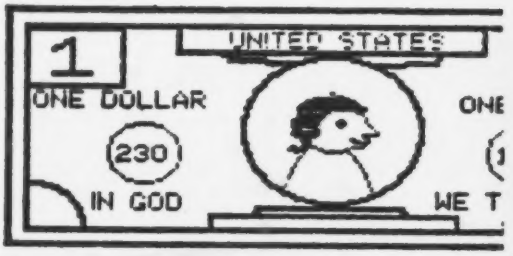
$$S = \begin{Bmatrix} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{Bmatrix}$$

EVENT	SUM	N(E)	N(E)/N(S)= P(E)
$E_2 = \{(1,1)\}$	2	1	1/36 ≐ .028
$E_3 = \{(1,2), (2,1)\}$	3	2	2/36 ≐ .056
$E_4 = \{(1,3), (2,2), (3,1)\}$	4	3	3/36 ≐ .083
$E_5 = \begin{Bmatrix} (1,4), (2,3), (3,2), \\ (4,1) \end{Bmatrix}$	5	4	4/36 ≐ .111
$E_6 = \begin{Bmatrix} (1,5), (2,4), (3,3), \\ (4,2), (5,1) \end{Bmatrix}$	6	5	5/36 ≐ .139
$E_7 = \begin{Bmatrix} (1,6), (2,5), (3,4), \\ (4,3), (5,2), (6,1) \end{Bmatrix}$	7	6	6/36 ≐ .167
$E_8 = \begin{Bmatrix} (2,6), (3,5), (4,4), \\ (5,3), (6,2) \end{Bmatrix}$	8	5	5/36 ≐ .139

E_9	= {(3,6), (4,5), (5,4), (6,3)}	9	4	4/36 ≐ .111
E_{10}	= {(4,6), (5,5), (6,4)}	10	3	3/36 ≐ .083
E_{11}	= {(5,6), (6,5)}	11	2	2/36 ≐ .056
E_{12}	= {(6,6)}	12	1	1/36 ≐ .028

Compare your results for the actual trials, computer simulation, and mathematical solution.

PAY YOUR BILL



EXPERIMENT: When Tom tries to collect $5 a week for delivering the daily paper, Mr. Bernoulli offers Tom a choice. He proposes that he either pay Tom $5 weekly or each week let Tom reach into a bag containing five $1 bills and one $10 bill and draw two bills. Being a doubter, Tom is suspicious that Mr. Bernoulli's scheme is a scam. Which method of payment should Tom choose?

ACTUAL TRIALS: Put five $1 bills and one $10 bill into a paper bag. (Monopoly money, bingo chips in two colors, and so on, can be used.) Draw two bills from the bag and record the sum (that is, write down either $2 or $11). Perform the experiment 52 times, tallying the results of $2 and $11. Add the total amount drawn in the two trials. Divide the sum by 52 to get the weekly average for one year.

COMPUTER SIMULATION: RUN PAY YOUR BILL

Suggestion: List the whole program by typing LIST and pressing the Return key. Determine which lines "draw the bills from the bag" (that is, generate the random numbers).

MATHEMATICAL SOLUTION:

$$S = \begin{Bmatrix} (T,1), (T,2), (T,3), (T,4), (T,5), \\ (1,2), (1,3), (1,4), (1,5), \\ (2,3), (2,4), (2,5), \\ (3,4), (3,5), \\ (4,5) \end{Bmatrix} \qquad N(S) = 15$$

Let E_{11} be the event of drawing \$11 and E_2 be the event of drawing \$2. Let T represent the \$10, let 1 represent one of the \$1 bills, let 2 represent the second \$1 bill,..., let 5 represent the last \$1 bill.

$$E_{11} = \{(T,1), (T,2), (T,3), (T,4), (T,5)\}$$

$$N(E_{11}) = 5$$

$$P(E_{11}) = N(E_{11})/N(S) = 5/15$$

$$E_2 = \begin{Bmatrix} (1,2), (1,3), (1,4), (1,5), (2,3), \\ (2,4), (2,5), (3,4), (3,5), (4,5) \end{Bmatrix}$$

$$N(E_2) = 10$$

$$P(E_2) = N(E_2)/N(S) = 10/15$$

EXPECTED VALUE = VALUE OF E_{11} X PROBABILITY OF E_{11} +
VALUE OF E_2 X PROBABILITY OF E_2

= \$11 X (5/15) + \$2 X (10/15) = \$5

Hence the offer was fair. (See Appendix A for the definition of expected value.)

Compare your results for the actual trials, computer simulation, and mathematical solution.

THE BIRTH MONTH PROBLEM

EXPERIMENT: What is the probability that in a group of five people at least two were born in the same month of the year?

ACTUAL TRIALS: Number twelve slips of paper from 1 to 12 and place them into a bag. Without looking, draw one slip from the bag, write down the number, and return the slip to the bag. Repeat until you have written down five numbers. If you had one or more repeats (for example, 2, 5, 10, 2, 3), the event was a "success." If no repeats occurred, the event was not successful. Repeat the experiment 50 (100? 200?) times. Calculate the relative frequency of success by dividing the number of successes by 50 (the number of trials).

Actual trials for this experiment can be made more interesting by selecting five students, asking them to name their birth months, and noting whether or not at least two have the same birth month. This is considered one trial. Repeat with several different groups of five students. A trial is considered a "success" if two or more in a group of five students have the same birth month. Find the relative frequency by dividing the number of successful trials by the total number of trials.

COMPUTER SIMULATION: RUN BIRTH PROBLEM

Suggestions:

1. On the first run, choose 50 trials (the same as the number of actual trials above).

2. LIST 10,110 and determine which lines choose the months of the year (generate the random numbers).

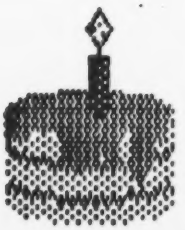
MATHEMATICAL SOLUTION: The probability that 2 people don't have the same birth month is 11/12. The probability that 2 out of 3 people don't have the same birth month is (11/12) X (10/12). The probability that 2 out of 4 people don't have the same birth month is (11/12) X (10/12) X (9/12). The probability that 2 out of 5 people don't have the same birth month is (11/12) X (10/12) X (9/12) X (8/12) = 0.38. The probability that 2 people out of 5 do have the same birth month is 1 - 0.38 = 0.62. (See Appendix A for the definition of complement of an event.)

Compare your results of actual trials, computer simulation, and mathematical solution.

The Birth Month Problem can be modified by changing the number of people in the group. The following table illustrates the probability that two or more people will have the same birth month in a group of n people. These mathematical solutions can be verified by running the program BIRTHMONTH CALCULATION, which is not a simulation. The program gives the mathematical solution for groups of n people, analogous to the mathematical solution presented above for a group of five people.

No. in Group	Probability That Two or More Have the Same Birth Month
2	.08
3	.23
4	.42
5	.61
6	.77
7	.88
8	.95
9	.98
10	.99

THE BIRTHDAY PROBLEM



EXPERIMENT: What is the probability that in a group of 23 people at least 2 were born on the same day of the year?

ACTUAL TRIALS: Number 365 slips of paper from 1 to 365 and put them into a bag. Without looking, draw one slip from the bag, write down the number, and return the slip to the bag. Repeat until you have written down 23 numbers. This is one trial. If you have one or more repeats, the trial was a "success." If no repeats occurred, the trial was not successful. Repeat the experiment 30 (50? 100?) times. Calculate the relative frequency of success by dividing the number of successes by 30 (the number of trials).

It is more interesting to run an actual trial by having 23 students name their birthdays. Ask each student to say "Stop!" if his or her birthday is named. When real people and birthdays are used, the number of actual trials is limited. However, the actual trials can be repeated if students are asked to commit themselves mentally and to report one birthday different from their own (e.g., the birthday of a brother, their mother).

COMPUTER SIMULATION: RUN BIRTH PROBLEM

Suggestions:

1. On the first run, choose 30 trials (the same as the number of actual trials above).

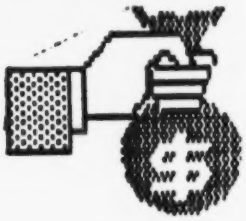
2. LIST 10,110 and determine which lines choose the days of the year (generate the random numbers).

MATHEMATICAL SOLUTION: The probability that 2 people don't have the same birthday is 364/365. The probability that 2 out of 3 people don't have the same birthday is (364/365) X (363/365). The probability that 2 out of 4 people don't have the same birthday is (364/365) X (363/365) X (362/365)...The probability that 2 out of 23 people don't have the same birthday is (364/365) X (363/365) X (362/365) X (361/365)... X (343/365) ≐ Ø.5Ø. The probability that at least 2 people out of 23 do have the same birthday is 1 - Ø.5Ø = Ø.5Ø. (See Appendix A for the definition of complement of an event.)

Compare your results for the actual trials, computer simulation, and mathematical solution.

The Birthday Problem can be modified by changing the number of people in the group. The following table illustrates the probability that two or more people in a group of n people will have the same birthday. These mathematical solutions can be verified by running the program BIRTHDAY CALCULATION, which is not a simulation. The program gives the mathematical solution for groups of n people, analogous to the mathematical solution presented above for a group of 23 people.

No. in Group	Probability That Two or More Have the Same Birthday
5	.Ø2
1Ø	.11
15	.25
2Ø	.41
23	.5Ø
25	.56
3Ø	.7Ø
35	.81
4Ø	.89
45	.94
5Ø	.97



CATCHING THE COUNTERFEITER

EXPERIMENT: A king suspects that his treasurer is robbing him by substituting counterfeit coins for gold ones. The coins are packed 25 to a bag, and, in fact, the treasurer has been placing one false coin in every bag. The king commands that he be brought 25 bags of gold. He will select one coin from each bag to be analyzed for metal content. What is the probability that the king will find a counterfeit coin?

ACTUAL TRIALS: Put 24 green bingo chips (or poker chips or slips of green paper) and 1 red bingo chip into a bag. Without looking, draw one chip from the bag, observe its color, and return it to the bag. Repeat this procedure 25 times or until the red chip is drawn. If you drew 25 green chips, put a tally on a paper next to the words "Escaped Detection." If you drew a red chip, put a tally on the paper next to the word "Detected." This is the end of one trial (not 25 trials). Perform 50 (100?) trials. Calculate the relative frequency of detection by dividing the number of tallies next to the word "Detected" by 50 (the number of trials).

COMPUTER SIMULATION: RUN CATCHING THE COUNTERFEITER

Suggestion: LIST the whole program and determine which line "draws the coin."

MATHEMATICAL SOLUTION: The probability of drawing the false coin from the first bag is 1/25. Hence the treasurer's probability of escaping

detection when the first bag is opened is 24/25. The probability of escaping detection for two bags is 24/25 X 24/25,..., and the probability of being safe through 25 bags is $(24/25)^{25} \doteq 0.36$. Hence the probability of being caught is 1 - 0.36 = 0.64.

Compare your results for the actual trials, computer simulation, and mathematical solution.

MEETING FOR LUNCH

EXPERIMENT: Two friends who have unpredictable lunch hours agree to meet for lunch at their favorite restaurant whenever possible. Neither wishes to eat alone and each dislikes waiting for the other, so they agree that--

1. each will arrive at a random time between 12:00 noon and 1:00 p.m.;

2. each will wait for the other for 15 minutes or until 1:00 p.m.

On a given day, what is the probability that the friends will meet for lunch?

ACTUAL TRIALS: Make a spinner like the one in figure 1. The pointer can be borrowed from another game or purchased at NCTM national conventions, or you can use a paper clip instead of an arrow pointer. One trial will consist of two spins. If the two spins stop within 15 minutes of each other, the trial is considered a success and a tally is recorded on a sheet of paper next to the words Met for Lunch. If the two spins do not stop within 15 minutes of each other, the trial

is not a success and a tally is recorded next to the words Did Not Meet for Lunch. Perform 100 trials (two spins per trial), tallying successful and unsuccessful outcomes. Total the number of successes (Met for Lunch) and divide by 100 to find the relative frequency of the friends meeting for lunch.

Fig. 1

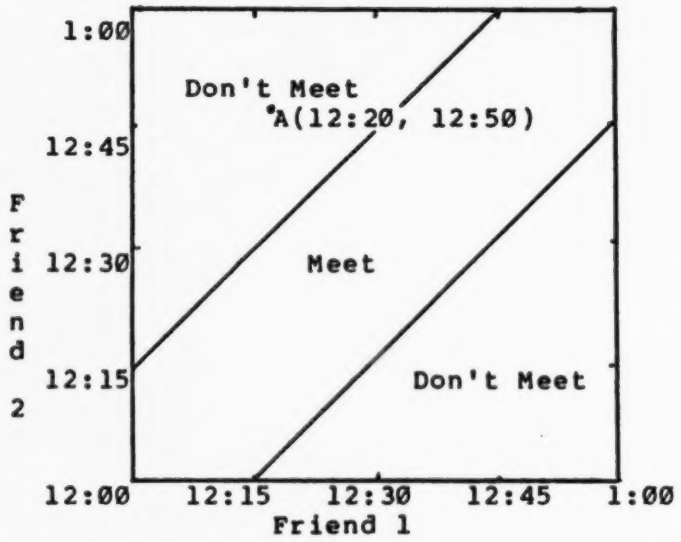
COMPUTER SIMULATION: RUN MEETING FOR LUNCH

Suggestion: LIST the whole program and determine which lines determine the arrival times of the two friends.

MATHEMATICAL SOLUTION: Consider the square in figure 2. The horizontal axis represents the arrival time between 12:00 noon and 1:00 p.m. of Friend 1. The vertical axis represents the arrival time of Friend 2. The interior of the square represents the arrival time of both friends on any particular day. For example, if Friend 1 arrives at 12:20 p.m. and Friend 2 arrives at 12:50 p.m., the ordered pair (12:20, 12:50), point A, represents the trial. This trial was unsuccessful, that is, the friends did not meet for lunch because their arrival times differed by more than 15 minutes (50 - 20 = 30 minutes). The interior of the square can be labeled Don't Meet and Meet, according to the difference between the two arrival times. The area of each of the corner triangles is 1/2 X base X height = 1/2 X 45 X 45 = 2025/2. The sum of the areas of the two triangles is 2025 square units. The area of the square is 60 X 60 = 3600 square units.

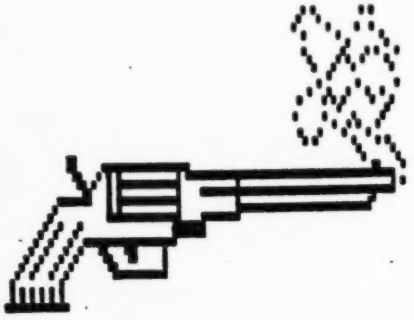
The area of Meet is the area of the square minus the areas of the triangles marked Don't Meet, that is, 3600 - 2025 = 1575 square units. The probability of the friends meeting for lunch is the area of the event (Meet) divided by the area of the sample space (the entire interior of the square) = 1575/3600 = 7/16 ≐ 0.44.

Fig. 2



Compare your results for the actual trials, computer simulation, and mathematical solution.

THE DUELLING MARKSMEN



<u>EXPERIMENT</u>: Two marksmen have a duel. They each have a 50% chance of hitting their target on each shot. What is the probability that the first man to shoot wins the duel?

<u>ACTUAL TRIALS</u>: Two students "duel" by tossing a coin to model the 50% chance of shooting each other. Record the results of each "duel" on a

piece of paper by putting a tally under the heading First Marksman or under the heading Second Marksman.

The first "marksman" tosses a coin. If the result is heads, the duel is over and a tally is put in the column First Marksman. If the result is tails, the first marksman has missed his man and it is the second marksman's turn to "shoot." The second marksman tosses the coin. If the result is heads, the trial is over and a tally is put in the column Second Marksman. If the result is tails, the second marksman has missed his man and it is the first marksman's turn to shoot. The first and second marksmen continue to take turns until one of them tosses a head, in which case one trial is complete and one tally is recorded in either the First Marksman column or the Second Marksman column.

Repeat the procedure until 50 (100?) trials have been performed. Calculate the relative frequency of success by the first marksman by dividing the number of tallies in the column First Marksman by 50 (the total number of trials).

COMPUTER SIMULATION: RUN THE DUELLING MARKSMEN

Suggestion: LIST 400,500. Find the lines that generate random numbers and explain how they determine whether the "shot" was a hit or a miss.

MATHEMATICAL SOLUTION: The probability that the first man to shoot wins the duel is 1/2. If he misses on the first shot, his probability of winning on his second shot is 1/2 X 1/2 X 1/2 = 1/8. [In order for the first man to win on his second shot, he would have to miss on his first shot (probability = 1/2), Marksman 2 would have to miss on his

first shot (probability = 1/2), and Marksman 1 would have to hit on his second shot (probability = 1/2).]

If Marksman 1 misses on his second shot, too, his probability of winning on his third shot is 1/2 X 1/2 X 1/2 X 1/2 X 1/2 = 1/32. The probability of Marksman 1 winning the duel is

$1/2 + (1/2 \times 1/2 \times 1/2) + (1/2 \times 1/2 \times 1/2 \times 1/2 \times 1/2) + \ldots =$

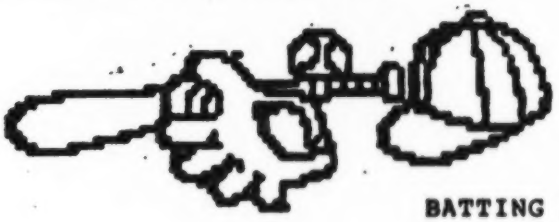
$1/2 + (1/2 \times 1/4) + (1/2 \times 1/16) + (1/2 \times 1/64) + \ldots =$

$1/2 + 1/2\,[1/4 + 1/16 + 1/64 + \ldots] =$

$1/2 + 1/2\,[1/4 + (1/4)^2 + (1/4)^3 + \ldots],$

which is a geometric series with first term of 1/2 and common ratio of 1/4. The sum of an infinite series $\sum_{i=1}^{\infty} a \times r^{i-1}$ is $a/(r-1)$, where a is the first term and r is the common ratio. Applying this formula, where a = 1/2 and r = 1/4, we find that the probability that Marksman 1 will win is $(1/2)/(1 - 1/4) = 2/3 \doteq 0.667$.

Compare your results for the actual trials, computer simulation, and mathematical solution.



BATTING PRACTICE

EXPERIMENT: How likely is a baseball player with batting average of 0.3 to get 3 or more hits if he is at bat 5 times in a game?

ACTUAL TRIALS: Put 3 red marbles and 7 black marbles into a bag. Draw out one marble, representing one time at bat. Record the color of the marble, then put it back into the bag. Repeat 5 times, representing 5 times at bat (one trial). If the total number of red

marbles is 3, 4, or 5, the trial is a success. Perform 50 (100?) trials. Calculate the relative frequency of success by dividing the number of successes by 50 (the number of trials).

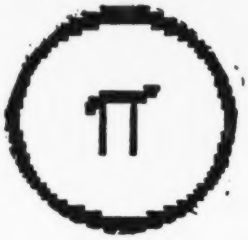
COMPUTER SIMULATION: RUN BATTING PRACTICE

Suggestion: LIST 600,700 and determine which line "hits" or "misses" the ball.

MATHEMATICAL SOLUTION: Suppose an experiment consists of n independent trials (see Appendix A for the definition of independent trials). If the event E is the same for each trial and its probability for a single trial is p, then the probability that the event will occur exactly r times is given by ${}_nC_r \times p^r \times (1 - p)^{n-r}$, where ${}_nC_r$ is the combination of n things taken r at a time and equals $n!/[(n-r)! \times r!]$. Applying this theorem to the Batting Practice problem, where n = 5 and r = 3, we find that the probability that the batter will get exactly 3 hits is given by ${}_5C_3 \times p^3 \times (1 - p)^{5-3}$. The probability that the batter will get exactly 4 hits is given by ${}_5C_4 \times p^4 \times (1 - p)^{5-4}$. The probability that the batter will get exactly 5 hits is ${}_5C_5 \times p^5 \times (1 - p)^{5-5}$. The probability of the batter getting 3 or more hits is the sum of the three expressions. Letting p = 0.3 and making the substitutions ${}_5C_3 = 10$, ${}_5C_4 = 5$, and ${}_5C_5 = 1$, we find that the answer is:

$10 \times (0.3)^3 \times (0.7)^2 + 5 \times (0.3)^4 \times (0.7)^1 + 1 \times (0.3)^5 \times (0.7)^0 \doteq 0.163$.

Compare your results for the actual trials, computer simulation, and mathematical solution.



MONTE CARLO PI

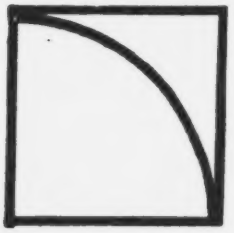
EXPERIMENT: Approximate the value of pi, using the Monte Carlo method.

ACTUAL TRIALS: On a piece of lightweight cardboard, draw a quarter circle inscribed in a square with side = 1 foot (see fig. 3). Tape this target on the center of a large piece of plywood. Ask student volunteers to throw 5 to 10 darts. The object is to throw the darts at the target and record on a tally sheet how many darts land inside the quarter circle and how many land inside the square but outside the quarter circle. Darts that land on the boundaries of one of these regions can be counted alternately in or out. Students can take turns throwing the darts and tallying the results until a total of 300 (500? 1000?) darts have been thrown. The area of the quarter circle is $1/4 \times \pi \times r^2 = 1/4 \times \pi \times 1 = \pi/4$.

If we assume that the darts hit the quarter circle in a random manner, the ratio of the number of darts that land within the quarter circle to the number that land anywhere within the square target should approximate the ratio of the area of the quarter circle to the area of the square, which is $\pi/4$ divided by 1. Hence 4 times that ratio is an approximation of π.

Divide the number of darts that land within the quarter circle by the number of darts that land anywhere within the square and multiply the result by 4 to determine the result of the actual trial.

Fig. 3

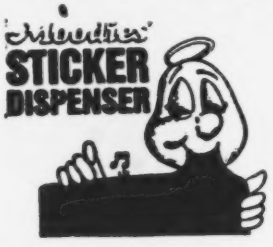


COMPUTER SIMULATION: RUN MONTE CARLO PI

Suggestion: LIST the whole program and determine which lines "throw the dart," that is, randomly generate the x and y coordinates of the points that are plotted on the target.

MATHEMATICAL SOLUTION: The area of the quarter circle is 1/4 X π X r^2 = π/4. The area of the square is s^2 = 1. The ratio of the two areas is π/4 divided by 1, which equals π/4. Four times this quantity is the value of π, which is approximately 3.1416.

Compare your results for the actual trial, computer simulation, and mathematical solution.



COLLECT ALL 3

THE CEREAL BOX PROBLEM

EXPERIMENT: Three different Moodies Musical Sticker Dispenser whistles are available in specially marked boxes of Kellogg's Rice Krispies. One prize appears in each box, and there is an equally likely chance of any one of the prizes being in any one box. How many boxes of cereal would you probably have to purchase before obtaining a complete set of three Musical Sticker Dispensers?

ACTUAL TRIALS: Number three slips of paper 1, 2, and 3, and put them into a bag. Draw one slip from the bag, observe the number, and record it on a tally sheet like the one below. Replace the slip. Draw again, observe the number, and record it on the tally sheet. Continue drawing and replacing slips until each of the numbers 1, 2, and 3 has been drawn. This constitutes one trial. For example, in Trial 1 recorded below, 7 slips were drawn before each number had been drawn once. The last number drawn in Trial 1 was 2. Repeat the experiment for a total of 25 trials. Calculate the average number of boxes of cereal purchased in order to get a complete set of whistles by dividing the total number of prizes by 25 (the number of trials).

Make a tally sheet like the following:

Trial	Prize 1	Prize 2	Prize 3	No. of Prizes
1	////	/	//	7
2	//	///	/	6
3				
.				
.				
.				
25	/	//	//	5
		Total Number of Prizes		

(This offer was actually conducted by the Kellogg Company, Battle Creek, Michigan, in 1985 and 1986. My son had to buy 9 specially marked boxes of Kellogg's Rice Krispies to collect a complete set of three different whistles. If a promotional offer such as this is running at the time your class is studying probability, take advantage of it as a real-life trial. The interest level of the students is terrific!)

COMPUTER SIMULATION: RUN THE CEREAL BOX PROBLEM

Suggestion: LIST 400,500 and determine which line puts the prize into the box (generates the random number).

MATHEMATICAL SOLUTION: (This solution [Keeping 1962] could be too difficult for many classes.) Suppose that we have collected x different prizes (x equals 1 or 2). The probability that the next box purchased contains one of the prizes we already have is x/3, and the probability that it has a new prize is 1 - x/3. The chance that two purchases will be necessary to get a new prize is x/3 X (1 - x/3). The probability that r purchases will be necessary to get a new prize is $(x/3)^{r-1}$ X (1 - x/3). Denoting the probability by p(r,x), the expectation of r for a given x is equal to

$$\sum_{r=1}^{\infty} r\,p(r,x) = \left(1-\frac{x}{3}\right)\sum_{r=1}^{\infty} r\left(\frac{x}{3}\right)^{r-1} = \left(1-\frac{x}{3}\right)\left[1 + 2\left(\frac{x}{3}\right) + 3\left(\frac{x}{3}\right)^{2} + \cdots\right]$$

$$= \frac{3}{3-x}.$$

The total expected number of purchases necessary to get all 3 prizes is therefore $\sum_{x=1}^{2} \frac{3}{3-x} = 3\left(\frac{1}{2} + \frac{1}{1}\right) = 3\left(\frac{3}{2}\right) = \frac{9}{2} = 4.5$ purchases.

V. APPENDIXES

APPENDIX A

Basic Probability Terms

In probability theory, problems are frequently called experiments or trials and are capable of being repeated over and over again. The result of an experiment is called an outcome. The set of all possible outcomes is the sample space, and an event is any subset of the sample space. If an experiment has s equally likely outcomes, and if e of these outcomes are considered successes, then the probability of success is defined to be the number of elements in the event E divided by the number of elements in the sample space: $P(E) = P(\text{success}) = n(E)/n(S) = e/s$, where n(E) equals the number of elements in E and n(S) equals the number of elements in S. Since any event E is a subset of the sample space S, then

$$\emptyset \leq n(E) \leq n(S); \quad \emptyset/n(S) \leq n(E)/n(S) \leq n(S)/n(S); \quad \emptyset \leq P(E) \leq 1.$$

If an outcome is certain to occur, its probability is 1; if an outcome is certain to fail, its probability is Ø. If E is an event and E' is the complement of the event, then $E \cap E' = \emptyset$ and $E \cup E' = S$, the whole sample space. $P(E) + P(E') = 1$ and $P(E) = 1 - P(E')$.

If an experiment is performed repeatedly n times under identical conditions and E is an event that occurs n_E times in those n repetitions of the experiment, then the relative frequency of E is denoted by $f_E(n) = n_E/n$ = (number of times E occurs)/(number of trials).

Example: Consider the experiment of tossing a fair coin three times. The set of all possible outcomes, the sample space, is S = {HHH, THH, HTH, HHT, TTH, THT, HTT, TTT}. One trial is the result of tossing the coin three times. If E is the event of tossing exactly 2 heads, then E = {THH, HTH, HHT}. E', the complement of E, is the event of tossing exactly 0, 1, or 3 heads. E' = {HHH, TTH, THT, HTT, TTT}. The probability of E = P(E) = n(E)/n(S) = 3/8; P(E') = 5/8; and P(E) + P(E') = 3/8 + 5/8 = 1.

Theorem. Let E and F be events. Then $P(E \cup F)$ = P(E occurs or F occurs) = $P(E) + P(F) - P(E \cap F)$.

Explanation: If E and F are subsets of sample space S, then $n(E \cup F) = n(E) + n(F) - n(E \cap F)$.

$$\frac{n(E \cup F)}{n(S)} = \frac{n(E) + n(F) - n(E \cap F)}{n(S)} = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(E \cap F)}{n(S)}$$

$$= P(E) + P(F) - P(E \cap F).$$

If E and F have no common elements (i.e., they are disjoint), then $E \cap F = \emptyset$ and $n(E \cap F) = \emptyset$. E and F cannot occur at the same time, and they are called mutually exclusive events. Hence we have the following:

Corollary. If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$.

Two events are independent if the occurrence of one event in no way influences the occurrence or nonoccurrence of the other event.

Theorem. Let E and F be independent events. Then the probability $P(E \cap F)$ = P(E occurs and F occurs) = P(E) X P(F).

Consider the example of tossing a fair coin three times. Let E be the event "tails on the first toss" and F be the event "tails on the second toss." E = {THH, TTH, THT, TTT}; n(E) = 4; P(E) = 4/8 = 1/2

F = {HTT, TTH, HTH, TTT} n(F) = 4 P(F) = 4/8 = 1/2

$E \cap F$ = {TTH, TTT} $n(E \cap F) = 2$ $P(E \cap F) = 2/8 = 1/4$

$P(E \cap F) = 1/4 = 1/2 \times 1/2 = P(E) \times P(F)$

Suppose the possible outcomes of an experiment are numbers, say a_1, a_2, a_3,..., a_n. If $P(\{a_i\})$ represents the probability that event $\{a_i\}$ occurs, then the expected value of the experiment is defined to be $a_1 \times P(\{a_1\}) + a_2 \times P(\{a_2\}) + \ldots + a_n \times P(\{a_n\})$. Example: See the mathematical solution to the experiment Pay Your Bill.

APPENDIX B

Brief History of Probability

The Chevalier de Mere, a wealthy French nobleman with a penchant for gambling, proposed to Blaise Pascal in 1654 the problem of distributing the stakes in an unfinished game of chance. When Pascal communicated the problem to Pierre de Fermat, the ensuing correspondence gave rise to a new field of mathematics--probability--which aroused considerable interest in the European scientific community (Berlinghoff 1968).

In 1713 Jacob Bernoulli published Ars Conjectandi, the first book devoted entirely to probability. The profound practical influence that probability would have on such fields as government, law, economics, and mortality was anticipated by Bernoulli by more than two centuries (Berlinghoff 1968).

Investigation of the history of probability is a fascinating research project for any secondary school or college student volunteer, integrating humanities, social science, and mathematics and familiarizing the student with such mathematical geniuses as De Moivre, Euler, Lagrange, Gauss, Laplace, Chebyschev, and Markov.

APPENDIX C

February 1986, Vol. 79, No. 2

Microcomputer Unit: Generating Random Numbers

By WILLIAM E. HAIGH, Northern State College, Aberdeen, SD 57401

Teacher's Guide

Grade level: 6–12

Completion time: One fifty-minute period on the computer

Objectives: Many concepts that are taught in a mathematics class can make use of programming skills to introduce or reinforce mathematics concepts. The generation of pseudorandom numbers is important for many simulations and games and deserves attention in a combined mathematics-programming unit. The primary objectives of this unit follow:

1. To investigate the RND function and determine the range of random numbers generated by making changes in the RND function
2. To investigate the role of the INT function in generating random integers in a given range
3. To give students an opportunity to use the microcomputer to discover how RND and INT can be used to generate a variety of random numbers
4. To allow students to generalize and make use of microcomputers to check their generalizations

Mathematics concepts that can be introduced or reinforced with the programming exercises include the recognition of integers and rational, irrational, and real numbers; the order properties and inequalities; open and closed intervals; computer limitations and precision; and the greatest-integer function. The mathematical concepts included are dependent on the type of mathematics class doing the programming exercises, but mathematical concepts and skills should be explicitly included.

Equipment: One or more microcomputers with monitors. (Apple, PET, TRS-80, and other microcomputers will work for this unit.)

Prerequisite experience: Students should have some experience with—

1. entering BASIC statements into a microcomputer;
2. using the RUN command to execute a program; and
3. changing statements in BASIC programs.

Directions: The implementation of this activity will depend on the number of microcomputers available. If you have only one microcomputer, you can have the students work through the activity in small groups (two to four students) on successive days. If you have a screen large enough for easy viewing, the entire class can participate. The ideal situation is to have enough computers for small groups to work on the activities at the same time. Appropriate mathematics concepts and skills should be introduced or reinforced during these activities. Additional ideas on random numbers are included in the "Microcomputer-assisted Discoveries" section of the December 1983 *Mathematics Teacher*.

Answers

1. Entries in table 1 should be greater than 0 and less than 1. With some microcomputer systems, "greater than" should be replaced by "greater than or equal to."

(Answers continued on page 136)

2. *a*) no, no
   b) 0, 1
3. Entries in table 2 should be greater than 0 and less than 5.
4. *a*) no, no
   b) 0, 5
5. *a*) 0, 8
   b) 0, 20
6. *a*) PRINT 12 * RND(1), or 12*RND on TI-99/4A
   b) No, infinitely many numbers occur between 0 and 12. (Note that because of the limited number of significant digits computers can handle, only a finite number of different values are possible.)
7. Entries in table 3 should be greater than 3 and less than 8.
8. *a*) In most cases the answers will be yes, yes, yes, and no.
   b) In most cases the answers will be yes, no, and no.
9. 3, 8
10. *a*) 0, 1, 2, 3, and 4
    b) 0, 4
    c) PRINT INT(8 * RND(1)).
    d) 5, 6, 7, and 8
11. PRINT INT(3 * RND(1)) + 9.
12. *a*) A determines the range of the integers to be generated.
    b) B determines the minimum value in the range of integers to be generated.
13. PRINT INT((S − R + 1) * RND(1)) + R. ■

MICROCOMPUTER UNIT:
GENERATING RANDOM NUMBERS

NAME _____

1. The following program will generate twenty random numbers. Enter the program into your microcomputer.

```
10 FOR I = 1 TO 20
20 PRINT RND(1),
30 NEXT I
40 END
```

Run the program three times and record in table 1 the largest and smallest numbers that the computer prints during each run. (Note that on some systems, TI-99/4A, for example, to get a different list you need to include the following line in your program: 5 RANDOMIZE.)

TABLE 1

Run	Largest Number	Smallest Number
First run		
Second run		
Third run		

2. *a*) Did you find any numbers larger than 1? _____
Smaller than 0? _____

b) After a large number of trials, the numbers generated would be greater than _____ and less than _____.

3. Change line 20 to the following:

```
20 PRINT 5 * RND(1),
```

Run the program three times and record in table 2 the largest and smallest numbers that are printed by the computer during each run.

TABLE 2

Run	Largest Number	Smallest Number
First run		
Second run		
Third run		

4. *a*) Are any numbers in table 2 larger than 5? _____
Smaller than 0? _____

b) The numbers generated are greater than _____ and less than _____.

From the *Mathematics Teacher*, February 1986

MICROCOMPUTER UNIT: GENERATING RANDOM NUMBERS—*Continued*

NAME _____

5. Making use of your responses to 4, complete the following statements:
   a) 8 • RND(1) will cause the computer to generate numbers that are greater than _____ and less than _____. Run the program with this change in line 20 and compare the results with your answers.
   b) 20 • RND(1) will cause the computer to generate numbers that are greater than _____ and less than _____. Run the program with this change in line 20 and compare the results with your answers.

6. *a*) On the line below, rewrite line 20 to cause the computer to generate numbers that are greater than 0 and less than 12.

   20 _____

   Make this change on the computer and run the program a few times to see if your answer is correct. Remember, the computer should generate and print numbers that are greater than 0 and less than 12.
   b) Do all the numbers "between" 0 and 12 occur? _____ Why? _____

_____

7. Change line 20 to

   20 PRINT 5 • RND(1) + 3,

   Run the program three times and record in table 3 the largest and smallest numbers that the computer prints during each run.

TABLE 3

Run	Largest Number	Smallest Number
First run		
Second run		
Third run		

8. *a*) Did you find any numbers larger than 5? _____ 6? _____ 7? _____ 8? _____
   b) Did you find any numbers smaller than 4? _____ 3? _____ 2? _____

9. Since 5 • RND(1) causes the computer to generate numbers greater than 0 and less than 5, 5 • RND(1) + 3 should cause the computer to generate numbers greater than _____ and less than _____. Verify your answers by referring to table 3.

From the *Mathematics Teacher*, February 1986

MICROCOMPUTER UNIT: GENERATING RANDOM NUMBERS—*Continued*

NAME _____

10. Change line 20 to the following:

```
20 PRINT INT(5 * RND(1)),
```

Run the program a few times and then answer the following questions.

a) List the different numbers that are generated by the computer.

_____

b) Recall that when line 20 was

```
20 PRINT 5 * RND(1),
```

the numbers generated were in decimal form and were greater than 0 and less than 5. Changing line 20 to

```
20 PRINT INT(5 * RND(1)),
```

causes the computer to generate integers that are greater than or equal to _____ and less than or equal to _____.

c) How should line 20 be changed to cause the computer to print the integers 0, 1, 2, 3, 4, 5, 6, and 7? Record your answer below:

20 _____

Run the program with this change to see if your idea is correct.

d) What integers will be generated by the computer when line 20 is written as follows?

```
20 PRINT INT(4 * RND(1)) + 5,
```

_____

Change line 20 to the form in question 10(*d*), run the program, and see if the results agree with the answer you gave in 10(*d*).

11. How should line 20 be changed to cause the computer to generate integers "between" 8 and 12? Record your answer below:

20 _____

Run the program with this change to see if your idea is correct.

12. In the statement

```
INT(A * RND(1)) + B,
```

a) How does the choice of A influence the random numbers generated?

_____

b) How does the choice of B influence the random numbers generated?

_____

13. Rewrite line 20 to generate a range of integers from R to S.

_____

From the *Mathematics Teacher*, February 1986

APPENDIX D

"Monte Carlo Method: Techniques of Simulation"

by Richard E. Wolfe, Department of Mathematics
Indiana University of Pennsylvania

The Monte Carlo method has its roots in the work of the Polish mathematician Stanislaw Ulam. His work began while he was grappling with the problem of what fraction of all games of solitaire could be completed satisfactorily to the last card. It was developed in the 1940s by Ulam and a group of mathematicians including John von Neumann, after Ulam had fled to the United States from his homeland because of the threat of Nazi Germany. Monte Carlo simulation was used to solve problems that arose in the development of the atomic bomb during the Manhattan Project at Los Alamos, New Mexico. Since then the availability of computers has enabled industry to use the Monte Carlo method to solve a wide range of problems, including the absorption of high-energy radiation, the computation of the thermodynamic properties of various systems, war games, and signal detection in the presence of random noise. Because it is so commonly used, students should be familiar with the techniques of this method, its uses, and its limitations.

The Monte Carlo method is a technique for solving problems using random outcomes of experiments, which in their simplest form involve such activities as tossing coins or rolling dice. It is used to determine the probability of an event when either--

1. there is no theory that adequately describes the event;
2. there is a theory that describes the event, but the computations involved become unwieldy if not downright

impossible.

The usual method of solving a probability problem is analysis. However, an analysis of interesting problems is often beyond the competence level of the secondary school mathematics student. A second method of solving probability problems is by experiment, sometimes not a very practical approach. Monte Carlo simulation becomes necessary when mathematical analysis is difficult or impossible and experimentation is expensive, time-consuming, or otherwise not feasible. It involves finding a model for the given problem. This model is physically diffrent and easier to operate but has the same mathematical characteristics as the original problem. The distinctive feature of the Monte Carlo method is the use of random devices such as dice, coins, spinners, or random numbers from tables, the telephone book, or the random-number generator of a computer. Almost any problem to determine the probability of success or failure or any problem to find an expected value can be solved by a suitable Monte Carlo simulation.

The theoretical basis for the Monte Carlo method is called the "law of large numbers," which states that as a simulation is run a larger and larger number of times, the simulated estimate

$$\frac{\text{number of successes}}{\text{number of runs}}$$

becomes approximately equal to the analytical probability.

The Monte Carlo method consists of four steps:

1. Identify an appropriate model. A random device must be selected that embodies the mathematical characteristics of the problem. The possible outcomes of the problem are

matched to outcomes of the random device that have the same probability. There are usually several ways of simulating the same problem.

2. Determine what will constitute a trial or "run" in the model, that is, one complete simulation of the problem, and the definition of a "successful" run.
3. Perform a large number of runs and record the results. In general, the larger the number of runs, the more accurate the result. For many problems, 100 runs are sufficient to provide a reasonably reliable estimate of the solution. To cut the error in half, the number of runs must be quadrupled.
4. Compute the simulated answer and interpret the results in terms of the given problem. For probability-of-success problems, the simulated estimate is

$$\frac{\text{number of successful runs}}{\text{number of runs}}.$$

For expected value problems, the expected value is

$$\frac{\text{sum of results from all individual runs}}{\text{number of runs}}.$$

See Collis (1982), Hinders (1981), Houser (1981), Lappan and Winter (1980), Travers and Gray (1981), Teaching Statistics and Probability (NCTM 1981), and Coughlin and Zitarelli (1984) in Annotated Bibliography and Other References.

APPENDIX E

Annotated Bibliography

Association of Mathematics Teachers of New Jersey. Activities Made to N-Joy, pp. 34-37. Glassboro, N.J.: The Association, 1985.

Four simulations written in BASIC: (1) The Coin Toss Simulation gives percentages of heads and tails. (2) Generating Frequency Distributions and Their Bar Graphs. (3) Playing Cards Simulation deals up to six hands, one card at a time; the user inputs the number of hands and the number of cards per hand. (4) Approximation of Pi by Simulation uses the Monte Carlo method to approximate pi. Graphics display the hits and misses of random points falling inside or outside a quarter circle inscribed in a unit square.

Camp, J. S. "Computer Simulations and Problem-Solving in Probability." Creative Computing, September-October 1978, pp. 69-71.

Presents BASIC computer simulations to solve three probability problems. The first determines the probability that in a family with exactly four children, all four are boys. The second involves collecting a complete set of 25 football pictures from packages of bubble gum (see Lappan and Winter [1980] and Travers and Gray [1981]). The third determines the reliability of an electrical system with five components arranged in parallel and series.

Carlson, Ronald J. "Buffon's Needle Problem on a Microcomputer." Mathematics Teacher 74 (November 1981): 638-40.

A computer simulation of Count de Buffon's needle drop estimation of pi. The BASIC computer program includes graphics of dropping needles on parallel lines one unit apart, recording the number of hits and misses, and estimating the value of pi.

Collis, Betty. "Simulation and the Microcomputer: An Approach to Teaching Probability." Mathematics Teacher 75 (October 1982): 584-87.

Description of an eighteen-lesson instructional unit for secondary students that uses classic probability situations, Monte Carlo simulation techniques, and a classroom microcomputer. Each student individually performs the actual experiment (problem), results are pooled, and inferences are made. A computer program simulating the same event is then run. Two examples, including their BASIC programs, are given: The Duelling Marksmen (see Hinders [1981]) and The Tower Game. References are given for nine other of the eighteen problems, some including theoretical solutions and graphics.

Ginther, John L., and William A. Ewbank. "Using a Microcomputer to Simulate the Birthday Coincidence Problem." Mathematics Teacher 75 (December 1982): 769-70.

The Birthday Problem (computing the probability that in a group of n people, two or more have the same birthday) is

approached in three ways: actual trials (three-digit numbers chosen from a table of random numbers or a telephone directory), computer simulation (written for a Commodore PET), and mathematical solution.

Haigh, William E. "Microcomputer Unit: Generating Random Numbers. Mathematics Teacher 79 (February 1986): 132-36.

A mathematics-programming unit to be completed by the student (grades 6-12) during one fifty-minute period on a microcomputer (e.g., Apple, PET, TRS-80). While completing the worksheet at the computer, the student discovers how the RND and INT functions can be used to generate a variety of random numbers. By the end of the period, the student can use the statement INT((S-R+1) * RND(1)) + R to generate integers from R to S.

----------. "Using Microcomputers to Solve Probability Problems." Mathematics Teacher 78 (February 1985): 124-26.

How microcomputers can be used to apply the Monte Carlo method to probability problems. For each of two problems, a program written in BASIC, a sample RUN, and a comparison of the computer and theoretical solutions are given. The Cookie Problem determines how many of 100 chocolate chip cookies will have no chips if exactly 150 chocolate chips are mixed well in the batter. The Bacteria Problem approximates the total number of bacteria in 100 sample drops of liquids containing bacteria that are randomly distributed in the liquids.

Hinders, Duane C. "Monte Carlo, Probability, Algebra, and Pi." Mathematics Teacher 74 (May 1981): 335-39.

Illustrates classroom generation of random numbers by flipping a coin, rolling a die, and tossing darts. The first method is applied to The Duelling Marksmen problem (see Collis [1982]) and the mathematical solution is also developed. A die is rolled for a Monte Carlo approach to the solution of a system of two simultaneous linear equations. The third Monte Carlo solution is an approximation of pi obtained by throwing darts at a target of four tangent circles.

Inhelder, William. "Solving Probability Problems Through Computer Simulation." In Teaching Statistics and Probability, 1981 Yearbook of the National Council of Teachers of Mathematics, pp. 220-24. Reston, Va.: NCTM, 1981.

Presents a BASIC program that simulates an ESP card experiment, such as that performed by J. B. Rhine at his Parapsychology Laboratory at Duke University.

Keating, B. "Simulations: Put the Real World in Your Computer." Creative Computing, November 1985, pp. 56-64.

Simulations as models of real systems, which allow the user to conduct trial-and-error experiments to predict the behavior of the actual system. Reasons for using simulations, difficulties with simulating, and simulation basics are discussed. Monte Carlo simulations of tossing a coin and bank customers arriving

and being served by an automatic teller machine are presented in BASIC. Other simulation languages and simulation software packages are described.

Kellogg, Howard M. "In All Probability, a Microcomputer." In Teaching Statistics and Probability, 1981 Yearbook of the National Council of Teachers of Mathematics, pp. 225-33. Reston, Va.: NCTM, 1981.

Presents BASIC programs (with graphics) for the three probability problems: (1) The Tontine, a life insurance policy (now illegal) for which the last surviving member of the group is awarded the entire contributed sum; (2) Monte Carlo estimation of area; and (3) Buffon's Needle (see Carlson [1981]).

Kimberling, Clark. "Generate Your Own Random Numbers." Mathematics Teacher 77 (February 1984): 118-23.

Discusses generating random numbers on an Apple computer. Equidistribution and runs tests written in BASIC are given.

Lappan, Glenda, and M. J. Winter. "Probability and Computers." Creative Computing, June 1985, pp. 70-72.

Four problems with their BASIC computer simulation programs and theoretical solutions. The problem of how to pay the weekly newspaper bill (see Walton [in press]) illustrates a discrete situation. The second problem, two friends meeting for lunch (see Walton [1986]), is continuous (i.e., the number of possible outcomes is infinite) and has a geometric theoretical solution. Problem three, randomly choosing two points on a unit circle and recording the distance between them, is continuous and uses trigonometry in the mathematical solution. The last problem, detecting counterfeit coins in a king's bags of gold (see Walton [1986]), is discrete but has a continuous extension leading to 1/e.

----------. "Probability Simulation in Middle School." Mathematics Teacher 73 (September 1980): 446-49.

The one-and-one foul shot in basketball is simulated by using a spinner with background labeled "Miss" or "Basket." Results are compared with theoretical expected frequencies. The problem of collecting a complete set of six Star Wars models by buying boxes of cereal (see Camp [1978]) is simulated using a spinner and by running a BASIC program. The theoretical expected number of boxes required to acquire a complete set is compared with the spinner and computer simulation results.

Merris, R. Introduction to Computer Mathematics, pp. 64-115. Rockville, Md.: Computer Science Press, 1985.

An integrated approach to probability and computer programming. Traditional topics from elementary probability theory (e.g., simulation, counting techniques, Pascal's triangle, expectation, conditional and binomial probability) are introduced with computer programming applications.

Piele, D. T. "How to Solve It with the Computer." Creative Computing, January 1981, pp. 142-51.
Introduction to problems that use the random number generator in versions of BASIC and the use of relative frequency as a measure of the probability of an event. Coin toss problems are discussed as well as the problem of determining the average number of tries necessary to write a bug-free program.

Teague, D. "Introduction to Probability and Simulation through Logo." In Focus on Statistics, pp. 317-44. Princeton, N.J.: Woodrow Wilson Fellowship Foundation Institute on High School Mathematics, 1984.
An introduction to the idea of probability as a relative frequency. Simulation programs written in Logo include estimating pi by the Monte Carlo method, a classical random walk problem, two friends meeting at the library (see Lappan and Winter [1985]), water-flow problems, breaking a stick into three pieces to form a triangle, soccer game simulations, and additional project ideas.

Travers, Kenneth J., and Kenneth G. Gray. "The Monte Carlo Method: A Fresh Approach to Teaching Probabilistic Concepts." Mathematics Teacher 74 (May 1981): 327-34.
The Bubble Gum Problem (see Lappan and Winter [1980]) and The Birthmonth Problem (see Ginther and Ewbank [1982]) are solved using a die or a spinner. A hand calculator that generates random numbers or a random number table is applied to The Shopping Mall Problem (what is the average distance between a given number of shoppers in a mall?). A fourth application to electronics involves two resistors connected in series and random numbers generated using a BASIC program. The theoretical answer for each problem is stated but not explained.

Walton, Karen D. "Microcomputers, Monte Carlo, and Mathematics." In Mathematics Teaching and Learning, 1986 Yearbook, pp. 63-68. University Park, Pa.: Pennsylvania Council of Teachers of Mathematics, 1986.
Two problems approached using the following format: statement of the problem, real-life trials, computer program and simulation, and mathematical solution. The problems are Detecting the Counterfeiter (see Lappan and Winter [1985]) and The Duelling Marksmen (see Collis [1982] and Hinders [1981]).

----------. "Teaching Probability: The Computer Is the Handmaiden of Mathematics." Journal of Computers in Mathematics and Science Teaching, Spring 1986, pp. 19-23.
Three problems approached using the format stated in Walton (1986). The problems are Throwing Two Dice, Paying the Newspaper Bill (see Lappan and Winter [1985]), and Monte Carlo Estimation of Pi (see Association of Mathematics Teachers of New Jersey [1985]). References to seven other problems are given.

APPENDIX F

Other References

Ahl, D. H., ed. More BASIC Computer Games. New York: Workman Publishing Co., 1979.

Berlinghoff, W. P. Mathematics, the Art of Reason. Boston: D.C. Heath & Co., 1968.

Coughlin, Raymond, and David E. Zitarelli. The Ascent of Mathematics. New York, N.Y.: McGraw-Hill Book Co., 1984.

Hoffer, A. R. Statistics and Information Organization: Mathematics Resource Project. Palo Alto, Calif.: Creative Publications, 1978.

Houser, Larry L. "Baseball Monte Carlo Style." Mathematics Teacher 74 (May 1981): 340-41.

Keeping, E. S. Introduction to Statistical Inference, p. 18. Princeton, N.J.: D. Van Nostrand Company, 1962.

Kellogg, H. M. "In All Probability, a Microcomputer." In Teaching Statistics and Probability, 1981 Yearbook of the National Council of Teachers of Mathematics, pp. 225-33. Reston, Va.: NCTM, 1981.

Rade, Lennart. "Random Digits and the Programmable Calculator." In Teaching Statistics and Probability, 1981 Yearbook of the National Council of Teachers of Mathematics, pp. 118-25. Reston, Va.: NCTM, 1981.

Shaughnessy, J. Michael. "Misconceptions of Probability: From Systematic Errors to Systematic Experiments and Decisions." In Teaching Statistics and Probability, 1981 Yearbook of the National Council of Teachers of Mathematics, pp. 90-100. Reston, Va.: NCTM, 1981.